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1985 J. Phys. A: Math. Gen. 18 1641

(http://iopscience.iop.org/0305-4470/18/10/019)

J. Phys. A: Math. Gen. 18 (1985) 1641-1656. Printed in Great Britain

Stochastic quantisation: stabilising quantum models

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Received 7 June 1984, in final form 27 November 1984

Abstract. Osterwalder-Schrader positivity is shown to be fulfilled for stochastically quantised lattice gauge theories, spin models and $P(\varphi)$ interactions bounded from below. Problems arising in the stochastic quantisation of bottomless $P(\varphi)$ models are discussed. A stochastic equation is derived to stabilise the quantum Einstein gravity. It is shown that the stochastic quantisation of the Yang-Mills theory leads to a well defined semiclassical expansion.

1. Introduction

Stochastic quantisation has been proposed by Parisi and Wu [1] mainly with the aim of avoiding the gauge fixing in continuum gauge theories. Subsequently numerous applications were found to the reduction of the number of degrees of freedom in the large N limit [2], [3] and to computer simulations [4]. On the perturbative level stochastic quantisation is equivalent to the conventional functional quantisation [5]. However, as pointed out by Greensite and Halpern in their inspiring paper [6] the stochastic method applies also to the theories with the classical action unbounded from below. This raises the question of whether such a stochastically quantised theory is relativistic. In particular, whether the Hamiltonian can be defined as an operator bounded from below. This will be the case, if the Osterwalder-Schrader (OS) positivity [7] is fulfilled. For this purpose the stochastic quantisation must be formulated in a way that ensures the OS positivity. Such a formulation is possible on the lattice.

In this paper we show first that the stochastic quantisation of lattice fields with values in a compact manifold M without a boundary is equivalent to the conventional average with respect to the equilibrium Gibbs measure. We discuss some problems concerning the OS positivity of bottomless $P(\varphi)$ models. We derive the path space measure for a stochastic process fulfilling the stochastic equation. As a direct outcome of the formalism of stochastic quantisation on manifolds we get a stochastic equation and a path space measure for a stochastic process on the manifold of Riemannian metrics. We show that the path space meaure for the stochastically quantised Yang-Mills theory leads to a well defined semiclassical expansion.

2. Stochastic quantisation on manifolds

Parisi and Wu [1] in their method of stochastic quantisation define a stochastic process as a solution of the Langevin equation. A solution of such an equation (with the white

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noise as a stochastic input) determines a Markov process (Prohorov and Rozanov [8]). The relation between the stochastic method of quantisation and the conventional Gibbs ensemble approach of Euclidean field theory (Glimm and Jaffe [14]) depends on various, sometimes inequivalent, characterisations of the Markov process.

The relation between these two methods of quantisation is simple in the case of lattice fields with values in a compact Riemannian manifold M without a boundary (e.g. gauge theory and σ-models on the lattice). Then, the stochastic quantisation leads to a Markov process on $M_n = X_1^n M$. A stochastic process $\xi(t)$ is defined by a probability measure on a set of functions (sample paths). In particular, the Markov process is characterised by its transition function $P(t, m, B)$ being the probability that $\xi(t)$, starting from $m \in M_n$, will be found in a set $B \subset M_n$. The Markov process is called stationary, if the correlation functions of $\xi(t)$ are invariant under translations in time. The Markov process is stationary, if it has a normalisable invariant measure μ on M_n ($\mu(M_n) = 1$) defined by

$$\int \mu(\mathrm{d}m) P(t, m, B) = \mu(B). \tag{2.1}$$

The asymptotic distribution appearing in the ergodic theory is determined by the invariant measure as

$$\lim_{t \to \infty} P(t, m, B) = \mu(B). \tag{2.2}$$

Conversely if we have the transition function $P(t, m_0, \mathrm{d}m) \equiv p(t, m_0, m)\, \mathrm{d}m$, where $\mathrm{d}m$ is the Riemannian volume element on M_n, and a normalisable invariant measure μ, then we can construct a stationary Markov process with μ as the initial distribution (see Prohorov and Rozanov [8] for these results).

The measure v on sample paths can be expressed by p as a limit (in the weak sense) of v_N where

$$\mathrm{d}v_N(m) = \left(\int \prod_{-N}^{N} \mathrm{d}m_s \exp(-L_N(m)) \right)^{-1} \prod_{-N}^{N} \mathrm{d}m_s \exp(-L_N(m)) \tag{2.3}$$

with

$$L_N(m) = -\sum_{s=-N}^{N} \ln p(\varepsilon, m_s, m_{s+1}). \tag{2.4}$$

The limit $N \to \infty$ in (2.3) does not depend on ε. Clearly, if the process ξ is stationary, then

$$\int F(\xi(t))\, \mathrm{d}v(\xi) = \int F(m) \mu(\mathrm{d}m). \tag{2.5}$$

If time t is continuous and A is the generator of the process ξ, defined by $(\partial/\partial t)p = -Ap$, then from (2.1) we have

$$A^* \mu = 0 \tag{2.6}$$

where A^* is the adjoint of A in $L^2(\mathrm{d}m)$.

For the purpose of stochastic quantisation we wish to construct a Markov process with the invariant measure μ. We apply the theory of Dirichlet forms in the formulation of Albeverio, Hoegh-Krohn and Streit [9]. Let A be the operator defined by the bilinear form

$$(f, Ah)=\tfrac{1}{2}\int \mu(\mathrm{d}m)\langle \mathrm{d}f, \mathrm{d}h\rangle_m \tag{2.7}$$

where $\langle\, ,\rangle$ is the Riemannian structure on TM_n^*.

Then, assuming $\mu(\mathrm{d}m)=[\int \mathrm{d}m \exp(-S(m))]^{-1}\,\mathrm{d}m \exp(-S(m))$, we find

$$A=-\tfrac{1}{2}\Delta_M-\tfrac{1}{2}X_b \tag{2.8}$$

where Δ_M is the Laplace-Beltrami operator on M_n and $X_b \in TM_n$ is the vector field with components

$$b(m)=-\nabla S(m). \tag{2.9}$$

By direct computation we can check that

$$A^*=-\tfrac{1}{2}\Delta_M+\tfrac{1}{2}X_b+\tfrac{1}{2}\nabla_\nu b^\nu \tag{2.10}$$

where ∇_ν denotes the covariant derivative, and that (2.6) is fulfilled.

It follows from the theory of diffusion processes (Ikeda and Watanabe [10]) that the process ξ defined by the generator A (2.8) is a solution of the stochastic equation (on a group manifold this equation has been derived previously by many authors, see e.g. [2], [3], [4])

$$\mathrm{d}\xi(t)=\tfrac{1}{2}b(\xi)\,\mathrm{d}t+\mathrm{d}\eta(t) \tag{2.11}$$

where $\eta(t)$ is the Brownian motion on M_n.

The Brownian motion on M_n has a simple description in terms of the (orthonormal) frame bundle $\mathrm{O}(M_n)$ (Ikeda and Watanabe [10]). Let $u(t)\in \mathrm{O}(M_n)$ be a solution of the stochastic equation

$$\mathrm{d}u(t)=E_a(u)\circ \mathrm{d}w^a(t) \tag{2.12}$$

where E_a is the canonical basis of horizontal vector fields in $\mathrm{TO}(M_n)$, w is the Wiener process ($\mathrm{d}w/\mathrm{d}t$ is the white noise) and $f\circ \mathrm{d}w$ denotes the symmetric Stratonovitch differential defined by the formula

$$\int_0^t f(\tau)\,\mathrm{d}w(\tau)=\lim_{N\to\infty}\sum_{i=0}^{N} f(\tfrac{1}{2}(\tau_i+\tau_{i+1}))(w(\tau_{i+1})-w(\tau_i))$$

with $\tau_{i+1}-\tau_i=t/N$. It can be shown that $f\circ \mathrm{d}w=f\,\mathrm{d}w+\tfrac{1}{2}\nabla f\,\mathrm{d}\tau$, where $f\,\mathrm{d}w=f(t)(w(t+\mathrm{d}t)-w(t))$ is the Itô differential.

Now, the projection of u on M_n defines the Brownian motion η on M_n. From (2.12), which describes a parallel displacement of the frame, we obtain

$$\mathrm{d}\eta_v=e_{va}(\eta)\circ \mathrm{d}w^a, \qquad \nabla_v e_{\mu a}\circ \mathrm{d}\eta^v=0 \tag{2.13}$$

where (e_a) is the frame on M_n ($e_{va}e_{\mu a}=g_{v\mu}$). In local coordinates we may write explicitly

$$\mathrm{d}\eta_v=e_{va}(\eta)\,\mathrm{d}w^a+\tfrac{1}{2}\Gamma^{\sigma}_{v\sigma}(\eta)\,\mathrm{d}t$$

where Γ are the Christoffel symbols.

The transition function for the process ξ can be computed from the Cameron-Martin-Girsanov formula, which may be considered as a representation of the Jacobian of the transformation $\eta \to \xi$. We have

$$p(t, m, m') = \int \mathrm{d}W^t_{(m,m')}(\eta) \exp\left(\tfrac{1}{2}\int_0^t e_{va}(\eta) b^v(\eta)\, \mathrm{d}w^a - \tfrac{1}{8}\int_0^t b_v b^v\, \mathrm{d}\tau\right)$$

$$= \exp(-\tfrac{1}{2}S(m') + \tfrac{1}{2}S(m)) \int \mathrm{d}W^t_{(m,m')}(\eta) \exp\left(-\int_0^t V(\eta)\, \mathrm{d}\tau\right) \tag{2.14}$$

where

$$V = \tfrac{1}{8} b_v b^v + \tfrac{1}{4}\nabla_v b^v \tag{2.15}$$

and $\mathrm{d}W^t_{(m,m')}$ is the Wiener measure on paths of the Brownian motion on M_n with $\eta(0) = m$ and $\eta(t) = m'$.

Equations (2.11), (2.13) and (2.14) can easily be checked by means of the Itô formula

$$\mathrm{d}f = \partial_v f\, \mathrm{d}\xi^v + \tfrac{1}{2}\partial_v \partial_\sigma f\, \mathrm{d}\xi^v\, \mathrm{d}\xi^\sigma$$

which allows us to show that $E[\mathrm{d}f] = -E[Af]\,\mathrm{d}t$ (see Simon [11]).

In order to check, if (2.2) is fulfilled, we note that (in Dirac notation from quantum mechanics)

$$\int \mathrm{d}W^t_{(m,m')}(\eta) \exp\left(-\int_0^t V(\eta)\, \mathrm{d}\tau\right) = \langle m | \exp - \tilde{A}t | m' \rangle$$

where $\tilde{A} = \exp(-\tfrac{1}{2}S)\; A \exp(\tfrac{1}{2}S) = -\tfrac{1}{2}\Delta_M + V$ and $\exp(-\tfrac{1}{2}S)$ is the ground state of $\tilde{A}$, hence

$$\langle m | \exp(-\tilde{A}t) | m' \rangle \to \exp(-\tfrac{1}{2}S(m)) \exp(-\tfrac{1}{2}S(m')) + \mathrm{O}(\exp - \varepsilon t). \tag{2.16}$$

The existence of the limit $N \to \infty$ in (2.3) and the equality (2.5) follow from the well known technique of the transfer matrix $T = \exp(-\varepsilon\tilde{A})$. If $\Omega(m)$ is the ground state of $\tilde{A}$, then

$$\int F(m)\Omega^2(m)\, \mathrm{d}m = \lim_{N\to\infty} (\langle m_0 | T^N | m_0 \rangle)^{-1} \langle m_0 | T^{N/2} F T^{N/2} | m_0 \rangle \tag{2.17}$$

as T^N projects on the ground state of A in the limit $N \to \infty$. As a consequence, if we take $\mathrm{d}m\, \Omega^2(m)$ as the Gibbs measure (see §4), then we get the equivalence with the standard Gibbs ensemble average.

The main statements formulated above remain valid for fields with values in a non-compact manifold provided that $\exp(-S)$ is integrable and the operator $\tilde{A}$ in (2.16) has a discrete spectrum. This is the case for $P(\varphi)$ models on a lattice with the interaction bounded from below. If these conditions are not fulfilled the problem of stochastic quantisation becomes much more involved. Solutions of the stochastic equations make sense only until a random explosion time $\tau(\eta)$ (see Elworthy [29]). The Cameron-Martin-Girsanov formula (2.14) still holds true, but the limits in (2.2), (2.3) and (2.17) may not exist. Hence, it may be impossible to eliminate the fictitious time variable entering the stochastic equations. We prove in § 3 that for lattice fields the limits (2.3) and (2.17) do exist (although (2.2) does not) also for bottomless $P(\varphi)$ models. In § 5 it is shown that the interaction (2.15) resulting from the stochastic quantisation is bounded from below for stochastically quantised Einstein gravity. Finally, in § 6 one-loop calculations are performed showing (in this approximation) the existence of the limit $t = N \to \infty$ in (2.3) for gauge fields in the continuum.

3. $P(\varphi)$ models with bottomless action

We now consider polynomial interactions U_ε such that U_ε is bounded from below, but $U_0=\lim_{\varepsilon\to 0} U_\varepsilon$ is not (e.g. $U=-x^4+\varepsilon x^6$). So, the action $S_\varepsilon(x)$, where $x=(x_1,\ldots,x_n)\in R^n$, has the form

$$S_\varepsilon(x)=-a\sum_{(k,r)} x_k x_r+\sum_k U_\varepsilon(x_k) \tag{3.1}$$

where k numbers the sites and (k,r) the bonds of a finite lattice. Then, the transition function $p(t,x,x')$ (equation (2.14)) is expressed by the kernel of the quantum mechanical n-particle Hamiltonian

$$\tilde{A}=-\tfrac{1}{2}\sum_k \Delta_k+V_\varepsilon(x) \tag{3.2}$$

where

$$V_\varepsilon(x)=\tfrac{1}{8}\sum_k \frac{\partial S_\varepsilon}{\partial x_k}\frac{\partial S_\varepsilon}{\partial x_k}-\tfrac{1}{4}\sum_k \frac{\partial^2 S_\varepsilon}{\partial x_k \partial x_k}. \tag{3.3}$$

It is known (Reed and Simon [12]) that if $V_\varepsilon(x)$ is bounded from below and $V_\varepsilon(x)\to\infty$, when $|x|\to\infty$, then $\tilde{A}$ has a purely discrete spectrum and the unique ground state. If moreover $\exp(-\frac{1}{2}S_\varepsilon)\in L^2(R^n)$, then $\exp(-\frac{1}{2}S_\varepsilon)$ is the ground state, because $\tilde{A}_\varepsilon\exp(-\frac{1}{2}S_\varepsilon)=0$ and $\tilde{A}_\varepsilon\geqslant 0$. Then, from (2.16) it follows that $p(t,x,x')\to\exp(-S_\varepsilon(x'))$, when $t\to\infty$. Equation (2.17) shows that we can construct the path space measure $\mathrm{d}v_\varepsilon(\xi)$ (equation (2.3)) corresponding to the stationary Markov process with the invariant measure $\mathrm{d}\mu_\varepsilon=(\int \mathrm{d}x\exp(-S_\varepsilon))^{-1}\,\mathrm{d}x\exp(-S_\varepsilon)$. Hence, the OS positivity

$$\int \mathrm{d}v_\varepsilon\, F\theta\bar{F}=\int \mathrm{d}\mu_\varepsilon\, F\theta\bar{F}\geqslant 0 \tag{3.4}$$

(where θ is a reflection of lattice sites with respect to a certain plane on a lattice) holds true for the v-expectation values, if it is true for the μ-expectation values (this requires $a\geqslant 0$ in (3.1) and the invariance of the finite lattice with respect to the reflection θ, see Fröhlich *et al* [13]).

Assume now that $S_\varepsilon\to S_0$ and $\nabla S_\varepsilon\to\nabla S_0$, that V_ε (equation (3.3)) is bounded from below uniformly in ε, and both $V_\varepsilon(x)$ and $V_0(x)$ tend to infinity when $|x|\to\infty$ (this is true, if $U_\varepsilon(x)$ is a polynomial). Then, $b_\varepsilon\to b_0$ in (2.11) and the transition function p_ε (equation (2.14)) converges to p_0. However, if $\exp(-S_0)$ is not integrable, then $p_0(t,x,x')$ has a trivial limit for $t\to\infty$ (so it does not fulfil (2.2)) as

$$p_0(t,x,x')\sim\exp(-\tfrac{1}{2}S_0(x')+\tfrac{1}{2}S_0(x))\Omega(x')\Omega(x)\exp(-E_0t) \tag{3.5}$$

for large t, where $E_0>0$ is the lowest eigenvalue of $\tilde{A}_0$ and $\Omega\neq\exp(-\frac{1}{2}S_0)$ is the corresponding eigenfunction (the ground state). Equation (3.5) can be proved by means of the transfer matrix method, because under our assumptions $\tilde{A}_0$ has a discrete spectrum and the unique ground state $\Omega\in L^2(R^n)$. The behaviour (3.5) could also be derived from the stochastic equation (2.11) with $b_0=-\nabla S_0$. When S_0 is unbounded from below, then the process ξ has a finite explosion time $\tau(\xi)$. In such a case

$$P_0(t,x,B)=P(\xi(t)\in B\,|\,\xi(0)=x,\ t<\tau(\xi))\sim P(t<\tau(\xi))\sim\exp(-E_0t)$$

for large t, where $P(\ |\ )$ denotes the conditional probability. In conclusion, the transition function p_0 does not determine a stationary Markov process, because the conditions (2.2)–(2.3) are not fulfilled.

Nevertheless, we can construct a stationary Markov process from the kernel of $\exp(-\tilde{A}_0 t)$. Let

$$p(t, x, x') = \Omega(x')\Omega^{-1}(x) \exp(E_0 t)\langle x|\exp(-\tilde{A}_0 t)|x'\rangle. \tag{3.6}$$

Then, it is easy to check that

$$P(t, x, B) = \int_B \mathrm{d}x'\, p(t, x, x') \tag{3.7}$$

is the transition function of a stationary Markov process with a finite invariant measure $\mathrm{d}\mu(x) = \mathrm{d}x\, \Omega^2(x)$, which coincides with the asymptotic distribution (2.2). Then, we can construct the path space measure v (2.3) corresponding to this stationary process, which we now express as a limit (in the weak sense) of v^t, where

$$\mathrm{d}v^t(\xi) = \left[\mathrm{d}W^{(-t,t)}_{(x,x')}(\xi) \exp\left(-\int_{-t}^{t} V_0(\xi(\tau))\,\mathrm{d}\tau\right)\right]^{-1} \times \mathrm{d}W^{(-t,t)}_{(x,x')}(\xi) \exp\left(-\int_{-t}^{t} V_0(\xi(\tau))\,\mathrm{d}\tau\right) \tag{3.8}$$

where $\mathrm{d}W^{(-t,t)}_{(x,x')}$ is the Wiener measure on paths with $\xi(-t) = x$ and $\xi(t) = x'$. From (2.17) we get

$$\int \mathrm{d}v\, F(\xi(t)) = \int \mathrm{d}x\, \Omega^2(x) F(x). \tag{3.9}$$

The measure $v = \lim v^t$ exists for any (bottomless) polynomial U. This measure has been proposed as a stabilised version of bottomless theories by Greensite and Halpern [6]. However, the stochastic process $\xi(\tau)$ defined by the measure v does not fulfil the stochastic equation for the initial theory S_0 (equation (3.1)), but instead a stochastic equation determined by the invariant measure $\mathrm{d}x\, \Omega^2$ (see [9])

$$\mathrm{d}\xi = \nabla \ln \Omega\, \mathrm{d}\tau + \mathrm{d}w. \tag{3.10}$$

$\ln \Omega^{-2}(x_1, \ldots, x_n)$ is in general a non-local function of x_k (i.e. not a sum over k). Hence, it would be hardly identifiable with a stabilised bottomless action, if not for the following remarkable properties of the measure v: (i) in a formal perturbation expansion of the LHS of (3.9) we get the perturbation expansion for bottomless S_0 theory as shown in reference [6]; (ii) $\mathrm{d}v^t$ is equal to $\lim_{\varepsilon\to 0} \mathrm{d}v^t_\varepsilon$, where v^t_ε is defined by (3.8) with V_0 replaced by V_ε (3.3) and $\exp(-S_\varepsilon) \in L^1(R^n)$ (this follows from the continuity of V_ε and its boundedness from below uniformly in ε). The property (i) may be considered as an indication that $\ln \Omega^{-2} - \frac{1}{2}\Sigma(x_k - x_m)^2$ behaves like $U(x)$ for small x. Moreover, from the results of Agmon and Lithner (see [15]) it follows that for large $|x|$ the 'effective action' $\ln \Omega^{-2}$ behaves like $|S_0(x)|$.

It still remains unclear whether theory (3.9) is OS positive. It would be so if we could show that v is a limit of v_ε fulfilling the OS inequality (3.4). We show that there is no v_ε, which could be constructed from S_ε of the form (3.1), if $x \in R$ (we owe this argument to S Kusuoka). In fact, let $\Omega_\varepsilon(x)$ be a continuous function, let $\mathrm{d}\mu_\varepsilon(x) = z_\varepsilon^{-1}\Omega_\varepsilon^2(x)\,\mathrm{d}x$ be the probability measure on R^n and $\Omega_\varepsilon \to \Omega_0 \notin L^2(R^n)$ uniformly on compact sets. Then, $z_\varepsilon^{-1} \to 0$, but $\int \mathrm{d}x\, \Omega_\varepsilon^2 F(x)$ remains finite for F with a bounded support, hence $v_\varepsilon \to 0$.

Remark: Let $S_\varepsilon(x) = S_0(x) + \varepsilon x^{2m}$, where S_0 is a bottomless polynomial. Then, for m large enough

$$V_\varepsilon(x) = V_0(x) + \varepsilon Q_\varepsilon(x),$$

where both V_0 and Q_ε are polynomials in x bounded from below and Q_ε is continuous in ε. It follows from the argument above and (3.9) that the limits $t \to \infty$ and $\varepsilon \to 0$ of the measure v_ε^t (3.8) are not interchangeable. Moreover, the perturbation $\varepsilon Q_\varepsilon$ in v_ε cannot be turned off (in contradistinction to the continuum $P(\varphi)_2$ models [14]). The fact that $Q_\varepsilon(x)$ although continuous in ε has minima tending to $\pm\infty$, when $\varepsilon \to 0$ seems to be responsible for the discontinuity of v_ε.

As there is no way to construct an OS measure on R^n such that $d\mu_\varepsilon \to dx\, \Omega^2(x)$, there remains to look for measures $d\mu_\varepsilon(x) = dx \exp(-S_\varepsilon(x))$ on I^n, where I is compact. In order to avoid the boundary problems in (2.6), it is useful to assume that $S(x)$ is periodic. Then, the OS action on $[-\pi, \pi]^n$, which is even and with the nearest-neighbour interactions has the form $(z > 0)$

$$S_\varepsilon(x) = -z\varepsilon^{-2} \sum_{\langle k,m \rangle} \cos(x_k - x_m) + \sum_k \sum_m c_m \cos m x_k. \tag{3.11}$$

With a proper choice of $c_n(\varepsilon)$ ($c_n(\varepsilon) \to \infty$ for $\varepsilon \to 0$) one can achieve the result that the formal 'low-temperature' limit of $\int d\mu_\varepsilon F$ will coincide with the theory (3.9). It is even possible that the low-temperature expansion is asymptotic to (3.9) as has been shown in the case of the plane rotator [28].

The main interest in bottomless actions concentrates on the negative coupling φ^4 in four dimensions (advocated some time ago by Symanzik [16]). The Lebowitz inequality, crucial in the proof of triviality of φ^4 [17], is violated in this model. The perturbative asymptotic freedom of the theory indicates ultraviolet stability. The negative coupling appears to be essential for Borel summability [18]. The OS positivity ensures the existence of the Hamiltonian bounded from below. However, it is hard to reconcile the asymptotic freedom, the Borel summability and the OS positivity. The Coleman argument [19] indicates that the negative coupling φ^4 should have effective action unbounded from below. The effective action seems to be positive for large fields in the model (3.9), because of the $\exp(-|S(x)|)$ behaviour, which we have mentioned before.

4. Lattice gauge theories

The formalism of § 2 finds applications to σ models and gauge theories, e.g.

$$S = -\beta \sum_{k,\mu} |z_{k+\mu} \bar{z}_k|^2 \tag{4.1}$$

where $\bar{z}_k z_k = 1$, z_k is a complex N vector and the sum is over all bonds (k, μ) of the lattice (this is the $CP(N-1)$ model [20] being a generalisation of the S^{N-1} model).

For the lattice gauge theories with the gauge group G we have [21]

$$S = -\beta \sum_P \operatorname{Tr} g_P = -\beta \sum_P \operatorname{Tr}(g_{k,\mu} g_{k+\mu,v} g_{k+v,-\mu} g_{k,-v}). \tag{4.2}$$

Here the sum is over all plaquettes P bounded by directed bonds (k, μ), $(k+\mu, v)$, $(k+v, -\mu)$ and $(k, -v)$, where μ denotes the direction and $g_{k,-\mu} = g_{k,\mu}^{-1}$.

For the weak coupling expansion in gauge theories [22], [23] it is useful to introduce an exponential parametrisation of the group element $g_{k,\mu} = \exp A_{k,\mu}$, where $A_{k,\mu} = \Sigma_a A^a_{k,\mu}\lambda_a$ and λ_a are the generators of the algebra (with $\operatorname{Tr}\lambda^a\lambda^b = -\delta^{ab}$). Then [22]

$$\begin{aligned} g(A+\delta A) &= g(A)(1+\delta A^a E_{ab}(A)\lambda^b) \\ &= (1+\lambda^a E_{ab}(A)\delta A^b)g(A) \end{aligned} \tag{4.3}$$

where

$$E_{ab}(A) = [(\exp A - 1)/A]_{ab}.$$

The Riemannian metric on G is defined by

$$\operatorname{Tr} \mathrm{d}g\, \mathrm{d}g^{-1} = g_{ab}\, \mathrm{d}A^a\, \mathrm{d}A^b. \tag{4.4}$$

From (4.3)-(4.4) we get

$$g_{ab}(A) = -E_{ac}(A)E^{\mathrm{T}}_{cb}(A) = 2[(\cosh A - 1)/A^2]_{ab}. \tag{4.5}$$

Using (4.3)-(4.5) and the formula

$$-\sum_a \operatorname{Tr}\mathscr{C}\lambda^a \operatorname{Tr}\mathscr{D}\lambda^a = \operatorname{Tr}\mathscr{C}\mathscr{D} - (1/N)\operatorname{Tr}\mathscr{C}\operatorname{Tr}\mathscr{D}$$

(N is the dimension of the representation) we get for the potential V (2.15)

$$\begin{aligned} V = -\frac{\beta^2}{8} \sum_{(k,\mu)} \sum_{P,P'\in(k,\mu)} [\operatorname{Tr} g_P g_{P'} - (1/N)\operatorname{Tr} g_P \operatorname{Tr} g_{P'}] \\ + \frac{\beta}{4} \sum_{(k,\mu)} \sum_{P\in(k,\mu)} \operatorname{Tr} g_P\left(\sum_a \lambda_a\lambda_a\right) \end{aligned} \tag{4.6}$$

for real representations (the sum is over plaquettes attached to the bond (k, μ)) and a real part of (4.6) for unitary representations.

The path space measure for the Markov process ξ on the group G could now be constructed as a limit $t \to \infty$ of $\mathrm{d}v^t$, where $\mathrm{d}v^t$ is defined like the measure in (3.8) with $\mathrm{d}W^{(-t,t)}_{(g,g')}$ being the Wiener measure for the Brownian motion on G. However, we prefer to use (2.3)-(2.4) and (2.14) to define the measure $\mathrm{d}g \exp(-\mathscr{L}(g))$ with $\mathscr{L}$ in a form analogous to the heat kernel action [24]. We cannot compute $p(\varepsilon, g, g')$ explicitly. However, we may take ε arbitrarily small to write $\mathscr{L}$ in the form

$$\mathscr{L}_{\mathrm{HK}} = -\sum_s \sum_{(k,\mu)} \ln p_0(\varepsilon, g_{k\mu}(s), g_{k\mu}(s+1)) + \sum_s \varepsilon V(g(s)) \tag{4.7}$$

where p_0 is the transition function for the Brownian motion on G (the heat kernel) or in a still simpler form

$$\mathscr{L} = -\sum_s \sum_{k\mu} \varepsilon^{-1} \operatorname{Tr}(g_{k\mu}(s) g^{-1}_{k\mu}(s+1)) + \sum_s \varepsilon V(g(s)). \tag{4.8}$$

It can be shown [25] that both $\mathrm{d}g \exp(-\mathscr{L}_{\mathrm{HK}})$ and $\mathrm{d}g \exp(-\mathscr{L})$ converge as $\varepsilon \to 0$ to the measure v defined by (3.8). (We could use formulae analogous to (4.7)-(4.8) also for $P(\varphi)$ models, $-\Sigma\, \varepsilon^{-1} \operatorname{Tr} g(s) g^{-1}_{(s+1)}$ is then replaced by $\frac{1}{2}\Sigma\, \varepsilon^{-1}(x_k(s) - x_k(s+1))^2$.) Only in the limit $\varepsilon \to 0$ do we recover the formula (2.5) and the OS positivity. The expression for $\mathrm{d}v = \mathrm{d}g \exp(-\mathscr{L})$ is the lattice approximation of the formal path space formula for the measure (3.8)

$$\mathrm{d}v = \Pi\, \mathrm{d}g \exp\left(-\frac{1}{2}\int \operatorname{Tr}\frac{\mathrm{d}g}{\mathrm{d}\tau}\frac{\mathrm{d}g^{-1}}{\mathrm{d}\tau}\,\mathrm{d}\tau - \int V(g(\tau))\,\mathrm{d}\tau\right). \tag{4.9}$$

There is a general rule for obtaining the formal path space measure (or more precisely the Lagrangian, see [26]) from the stochastic equation (we apply this rule in § 5). We express the free Lagrangian $L=\frac{1}{2}\int(\mathrm{d}W_a/\mathrm{d}\tau)(\mathrm{d}W_a/\mathrm{d}\tau)\,\mathrm{d}\tau$ for the Wiener process by the Brownian motion η on the manifold using the stochastic equation (2.13) (this is like the Nicolai mapping [27])

$$L=\frac{1}{2}\sum_a\int e_{a\mu}e_{a\upsilon}\frac{\mathrm{d}\eta^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\eta^\upsilon}{\mathrm{d}\tau}\,\mathrm{d}\tau=\frac{1}{2}\int g_{\mu\upsilon}(\eta)\frac{\mathrm{d}\eta^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}\eta^\upsilon}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{4.10}$$

The action $\mathscr{L}$ (4.6) and (4.8) could be applied for approximate calculations as an alternative to the conventional methods. It can easily be seen that at high temperatures β^{-1} (strong coupling) we get the Wilson area law and the mass gap. However, we expect that the formal [22], [23] low-temperature (weak coupling) expansions in the $\mathscr{L}$-theory and in the S-theory behave in a different way. Namely, the Laplace method for $S_J=S+Jg$ is not applicable (there is no minimum), but it seems to be applicable in the stochastic version, i.e. $\mathscr{L}(S_J)$ does have a minimum. We shall show this in a formal continuum limit (i.e. when $\exp A$ is expanded in A, the difference operators are replaced by differential operators and the non-classical terms [22] are neglected) in § 6. This observation inspires our aspiration for a rigorous approach to the low-temperature expansion in the $\mathscr{L}$-model of lattice gauge theory (see [28] for some results in this direction).

5. Stochastic quantisation of gravity

In the previous sections we could prove that a stationary Markov process ξ existed with the invariant measure μ owing to our assumption that M was finite dimensional (this corresponds to the field theory on a lattice). The theory of stochastic processes on infinite dimensional spaces is also well developed and already has found applications to quantum field theory [9]. Also the theory of stochastic processes on Hilbert manifolds has a sound mathematical basis (see e.g. [29], [30]). The stochastic quantisation of field theory needs distribution valued processes and can be formulated in the Hilbert space only through regularisation (this was the reason to use the lattice). Then, for some measures μ on Hilbert spaces (including the perturbed Wiener measure) a Markov process has been constructed with μ as an invariant measure (see [31], [32]). These results encourage us to treat the stochastic equations (2.11)-(2.13) in the Hilbert manifold $\mathscr{M}$ of all Riemannian structures on M [33].

The manifold $\mathscr{M}$ is a set of all sections in S^2T, the bundle of symmetric 2-tensors, which induce positive definite scalar product on each tangent space T_mM. The tangent space $T_g\mathscr{M}$ to $\mathscr{M}$ at $g\in\mathscr{M}$ is a subspace of S^2T. Since each $g\in\mathscr{M}$ is a Riemannian structure on TM, therefore it induces such a structure on $T_g\mathscr{M}\subset S^2T$ (for precise definitions and proofs we refer to [33]). Now let e_{kn} be an orthonormal frame at g i.e. an orthonormal base of vector fields in $T_g\mathscr{M}$. The collection of all orthonormal frames at all $g\in\mathscr{M}$ forms the frame bundle $\mathrm{O}(\mathscr{M})$. $\mathrm{O}(\mathscr{M})$ is a Riemannian manifold with the Levi-Civita connection Γ defined on it. Let E_{kn} be a base of horizontal vector fields in $\mathrm{O}(\mathscr{M})$. Then, we may define the Brownian motion on $\mathrm{O}(\mathscr{M})$ as a solution of a stochastic equation being a generalisation of (2.12):

$$\mathrm{d}\chi_\tau=E_{kn}(\chi)\circ\mathrm{d}W_\tau^{kn} \tag{5.1}$$

where W_τ^{kn} is the Wiener process indexed by the Hilbert space $S^2L^2(R^d)$ of square integrable 2-tensors on R^d, i.e.

$$E[(W_\tau, f)(W_{\tau'}, f')] = \min(\tau, \tau')(f, f') \tag{5.2}$$

with $(f, f') = \int \mathrm{d}x\, f^{kn}(x) f'_{kn}(x)$.

The Brownian motion h_τ on $\mathcal{M}$ can be defined as a projection of χ_τ on $\mathcal{M}$. We now define the drift term

$$b = -\mathscr{D}S \tag{5.3}$$

where $\mathscr{D}$ is the Fréchet derivative, i.e. $F(h+\varepsilon f) - F(h) = \varepsilon\langle \mathscr{D}F, f\rangle_g + O(\varepsilon^2)$ and $\langle\ \rangle_g$ is the Riemannian structure in $T_g\mathcal{M}$. Now, the stochastic equation for quantum gravity reads

$$\mathrm{d}g_\tau = \tfrac{1}{2} b\, \mathrm{d}\tau + \mathrm{d}h_\tau. \tag{5.4}$$

We may define the generator A of the process g_τ

$$(Af)(g_0) = \lim_{\tau\to 0} \tau^{-1}(E[f(g_\tau)] - f(g_0)). \tag{5.5}$$

We expect that, like in the finite dimensional case, in a certain sense

$$(f, Af') = \int_{\mathcal{M}} \mathrm{d}g \exp(-S(g))(\delta f, \delta f')_g \tag{5.6}$$

where $(\ )_g$ is the Riemannian structure on $T\mathcal{M}^*$ and $\mathrm{d}g \exp(-S(g))$ is to be the invariant measure for the Markov stochastic process g_τ.

In local coordinates $(x^\rho, g^{\mu\upsilon}(x))$ of $\mathcal{M}$ the orthonormal frame e^{kn} in $T_g\mathcal{M}$ defined by $\langle e_{kl}, e_{mn}\rangle_g = \delta_{km}\delta_{ln}$ can be chosen in the form

$$(e_{kl})^{\mu\upsilon}(x) = g^{-1/4}(x) e_k^\mu(x) e_l^\upsilon(x) \tag{5.7}$$

where $e_k{}^\mu(x) e_k{}^\upsilon(x) = g^{\mu\upsilon}(x)$ and $g = \det(g^{\mu\upsilon})$. Then, (5.1) projected on $\mathcal{M}$ has a form analogous to (2.13) (see [10], [34] for a detailed discussion of (2.12)-(2.13))

$$\begin{aligned} &\mathrm{d}g_\tau^{\mu\upsilon}(x) = e_{kl}^{\mu\upsilon}(g_\tau(x)) \circ \mathrm{d}W_\tau^{kl}(x) \\ &\nabla_{\rho\sigma} e_{kl}^{\mu\upsilon}(g_\tau(x)) \circ \mathrm{d}g_\tau^{\rho\sigma}(x) = 0 \end{aligned} \tag{5.8}$$

where

$$\nabla_{\rho\sigma} e_{kl}^{\mu\upsilon} = \partial e_{kl}^{\mu\upsilon}/\partial g^{\rho\sigma} + \Gamma^{(\mu\upsilon)}_{(\rho\sigma)(\alpha\beta)} e_{kl}^{\alpha\beta} \tag{5.9}$$

and

$$\begin{aligned} \Gamma^{(\mu\upsilon)}_{(\rho\sigma)(\alpha\beta)} = {} & g^{1/2}(\partial/\partial g^{\rho\beta})(g^{\mu\sigma}g^{\upsilon\alpha}g^{-1/2}) - g^{1/2}(\partial/\partial g^{\alpha\beta})(g^{\mu\rho}g^{\upsilon\sigma}g^{-1/2}) \\ & - g^{1/2}(\partial/\partial g^{\rho\sigma})(g^{\mu\alpha}g^{\upsilon\beta}g^{-1/2}). \end{aligned} \tag{5.10}$$

If we express the Stratonovitch differential ($f \circ \mathrm{d}W = f\,\mathrm{d}W + \frac{1}{2}\mathrm{d}f\,\mathrm{d}W$) by the Itô differential ($f\,\mathrm{d}W$), then (5.4) for g_τ has a singular form

$$\mathrm{d}g_\tau^{\mu\upsilon}(x) = \tfrac{1}{2} b^{\mu\upsilon}(g_\tau(x))\, \mathrm{d}\tau + \tfrac{1}{2}\Gamma^{\mu\upsilon}(g_\tau(x))\, \mathrm{d}\tau + g_\tau^{-1/4}(x) e_k^\mu(x) e_l^\upsilon(x)\, \mathrm{d}W_\tau^{kl}(x) \tag{5.11}$$

where $\Gamma^{\mu\upsilon} = \Gamma^{(\mu\upsilon)}_{(\rho\sigma)(\alpha\beta)} g^{\rho\alpha} g^{\sigma\beta} \delta_g(0)$ and $\int f(m)\delta_g(m, m') = f(m')$.

The singular factor $\delta_g(0)$ comes from the functional derivative $\delta g(m)/\delta g(m)$ resulting from the term $\mathrm{d}f\,\mathrm{d}W$. Equation (5.11) is useful, if we wish to check that (i) the operator (5.6) is the generator of the stochastic process g_τ, (ii) $A^*\,\mathrm{e}^{-S}=0$, i.e. e^{-S} fulfills the equation for the invariant measure, (iii) the transition function $P(t, g, g')$ is determined by the formula (2.14) with

$$V=\tfrac{1}{8}\int \mathrm{d}x\, g(x)^{1/2} b^{\mu\upsilon}(g) b_{\mu\upsilon}(g)+\tfrac{1}{4}\int \mathrm{d}x\, g(x)^{1/2}\delta_g(0)\nabla_{\mu\upsilon} b^{\mu\upsilon}(g) \tag{5.12}$$

and $\mathrm{d}W$ a 'Wiener measure on $\mathcal{M}$'.

Checking the points (i)-(iii) is just a repetition of the calculations from § 2 with functional derivatives replacing the ordinary derivatives (with $(\,,\,)_g=\int \mathrm{d}x(g(x))^{1/2}\langle\,,\rangle_{g(x)}$ in (5.6)). However, in order to really construct the process g_τ we have to construct the 'Wiener measure on $\mathcal{M}$'. Clearly, this is an extremely difficult problem (see, however, [30]). We can give only a formal expression for this measure. We follow the prescription from the previous section applied for the derivation of (4.9). Now, from (5.8)

$$\begin{aligned} L&=\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}\tau \frac{\mathrm{d}W_\tau^{kl}(x)}{\mathrm{d}\tau}\frac{\mathrm{d}W_\tau^{kl}(x)}{\mathrm{d}\tau}\\ &=\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}\tau\,(g_\tau(x))^{1/2} e_{\mu k}(x,\tau)e_{\upsilon l}(x,\tau)e_{\rho k}(x,\tau)e_{\sigma l}(x,\tau)\frac{\mathrm{d}g_\tau^{\mu\upsilon}(x)}{\mathrm{d}\tau}\frac{\mathrm{d}g_\tau^{\rho\sigma}(x)}{\mathrm{d}\tau}. \end{aligned} \tag{5.13}$$

Hence, the path space measure for g_τ has the form $\mathrm{d}g\exp[-\mathcal{L}(g)]$, where

$$\mathcal{L}(g)=\tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}\tau\,(g(x,\tau))^{1/2} g_{\mu\rho}(x,\tau)g_{\upsilon\sigma}(x,\tau)\frac{\mathrm{d}g^{\mu\upsilon}(x,\tau)}{\mathrm{d}\tau}\frac{\mathrm{d}g^{\rho\sigma}(x,\tau)}{\mathrm{d}\tau}+\int \mathrm{d}\tau\, V(g_\tau). \tag{5.14}$$

For the Einstein action $S=\int \mathrm{d}x\, g(x)^{1/2}R$ we get

$$V=\tfrac{1}{8}\int \mathrm{d}x\, g(x,\tau)^{1/2}R_{\mu\upsilon}R^{\mu\upsilon}+\tfrac{9}{8}\int \mathrm{d}x\, g(x,\tau)^{1/2}\delta_g(0)R+\text{surface terms}. \tag{5.15}$$

The path space action $\mathcal{L}$ is *bounded from below.* In such a case the functional integral is well defined after ultraviolet regularisation (the regularisation should preserve the invariance under the general coordinate transformations). The computations in § 6 show that if $\mathcal{M}^{-1}(x,y)$ is the propagator of the conventional quantum Einstein gravity (Hawking [43]), then the propagator resulting from stochastic quantisation, as the two-point function of the metric tensor g, has the form $|\mathcal{M}|^{-1}(x,y)$ (in spite of the formal resemblance to the R^2 theory with the p^{-4} propagator). Both propagators $\mathcal{M}^{-1}$ (having negative eigenvalues) as well as $|\mathcal{M}|^{-1}$ fail to fulfil the OS positivity. We do not find this situation disastrous. It is known, that in spite of the negative eigenvalues of $\mathcal{M}$ in the Euclidean theory, a unitary S-matrix can be defined in the one-loop quantum gravity for some physical scattering processes. The Wightman functions of the metric tensor g may have no physical meaning. Only some functionals of g will have a physical interpretation. For such functionals the OS positivity should be fulfilled, if the energy of physical states is to be bounded from below. Then we expect the formal perturbation series of both stochastically and conventionally quantised theories to coincide.

6. One-loop effective action

We now consider the continuum Yang–Mills theory. It is possible to write down the stochastic equation in the global form on the manifold of all connections (see [36], the formalism of Asorey and Mitter requires the knowledge of the physical ground state). We shall treat the continuum Yang–Mills theory as a formal continuum limit of the lattice theory as discussed at the end of § 4. The stochastic equation is

$$\mathrm{d}A(\tau, x) = \tfrac{1}{2}b(A)\,\mathrm{d}\tau + \mathrm{d}W_\tau(x) \tag{6.1}$$

where

$$b^a_\mu(A) = -\delta S(A)/\delta A^a_\mu(x). \tag{6.2}$$

The transition function (2.14) now has the form

$$P(A_t \in \Phi \mid A_0 = B) = \exp(\tfrac{1}{2}S(B)) \int_\Phi \mathrm{d}W_B(A(\cdot))\exp(-\tfrac{1}{2}S(A_t))\exp\left(-\int_0^t V(A_\tau)\,\mathrm{d}\tau\right) \tag{6.3}$$

where $\mathrm{d}W_B$ is the Wiener measure, i.e. the Gaussian measure with mean $B^{a\mu}(x)$ and covariance

$$E[(A^{a\mu}_t(x) - B^{a\mu}(x))(A^{b\nu}_{t'}(x') - B^{b\nu}(x'))] = \delta^{\mu\nu}\delta^{ab}\min(t, t')\delta(x - x')$$

and

$$V(A) = \tfrac{1}{8}\int b^a_\mu(A)b^a_\mu(A) + \tfrac{1}{4}\int \delta b^a_\mu/\delta A^a_\mu(x). \tag{6.4}$$

Equation (6.3) is well defined, if S and V are regularised. From the transition function we can construct the path space measure v. We construct first v' the path space measure on periodic paths $A_{-t} = A_t$ and then take the limit $t \to \infty$. Taking as the initial distribution the invariant measure $\mathrm{d}A\exp(-S(A))$ we get the following formula for v'

$$\begin{aligned}\mathrm{d}v'(A(\cdot)) &= \mathrm{d}A(\tau, x)\exp(-\mathscr{L}(A))\\ &= \mathrm{d}A(x)\exp(-S(A))\,\mathrm{d}W^{(-t,t)}_{(A,A)}(A(\cdot))\exp\left(-\int_{-t}^t V(A_\tau)\,\mathrm{d}\tau\right).\end{aligned} \tag{6.5}$$

Let

$$\hbar S(A) = \tfrac{1}{4}F^2(A) + JA. \tag{6.6}$$

We write

$$A(\tau, x) = B(x) + \sqrt{\hbar}\,Q(\tau, x) \tag{6.7}$$

where $Q(-t, x) = Q(+t, x) = 0$ and choose B to fulfil the classical equations $\delta S/\delta B = 0$. Then, we have to fix the gauge. We do this in the standard way (the stochastic quantisation offers new methods, see [37]) inserting the background gauge condition $\delta(\nabla^B_\mu Q)$. Then, expanding $\mathscr{L}$ around B we get

$$\mathscr{L}(B + Q) = S(B) + \tfrac{1}{2}Q(-\mathrm{d}^2/\mathrm{d}\tau^2 + \mathscr{M}^2)Q + \mathrm{O}(\hbar) \tag{6.8}$$

where

$$2\mathscr{M}_{\mu\nu} = -\nabla^B_\rho\nabla^B_\rho\delta_{\mu\nu} + 2\mathrm{i}F_{\mu\nu}(B). \tag{6.9}$$

Remark. We have put $\delta(0)\operatorname{Tr}\mathcal{M}(x,x)=0$ (the last term in (6.4)), because there is no local, gauge and Euclidean invariant expression of second order in A.

The main point we wish to emphasise in this section is the following: the second variation of $\mathcal{L}$ is always non-negative and for a class of sources J strictly positive, i.e. $\mathcal{L}$ has in fact a minimum (this is not true for $S(A)=\frac{1}{4}F^2(A)+JA$, see [38], [39]). In such a case the Laplace method can be applied leading to a well defined semiclassical expansion. Performing the Gaussian integral over Q and neglecting higher orders in $\hbar$ we get the one-loop formula [19] for the effective action

$$\Gamma_t(B)=\tfrac{1}{4}F^2(B)-\operatorname{Tr}\ln(-\nabla^B\nabla^B)+\tfrac{1}{2}\operatorname{Tr}\ln(-\mathrm{d}^2/\mathrm{d}\tau^2+\mathcal{M}^2). \tag{6.10}$$

The second term comes from the Faddeev-Popov determinant. The effective action Γ_t depends on t, because we have imposed periodic boundary conditions for the operator $-\mathrm{d}^2/\mathrm{d}\tau^2$ on the interval $[-t,t]$. There remains to perform the limit $t\to\infty$. For this purpose the trace over the eigenvalues $\pi^2n^2t^{-2}$ of $-\mathrm{d}^2/\mathrm{d}\tau^2$ has to be computed. We get (cf with similar computations in [40], here $\operatorname{Tr}\mathcal{M}=0$)

$$\begin{aligned}
&\operatorname{Tr}\ln(-\mathrm{d}^2/\mathrm{d}\tau^2+\mathcal{M}^2)\\
&\quad=-\operatorname{Tr}\int_0^\infty \mathrm{d}x\left[\sum_{n=0}^{\infty}(\pi^2n^2/t^2+\mathcal{M}^2+x)^{-1}\right]\\
&\quad=\tfrac{1}{2}\operatorname{Tr}\ln\mathcal{M}^2-2t^2\operatorname{Tr}\int \mathrm{d}x\left(\frac{\coth 2t(\mathcal{M}^2+x)^{1/2}}{2t(\mathcal{M}^2+x)^{1/2}}\right)\\
&\quad=\tfrac{1}{2}\operatorname{Tr}\ln\mathcal{M}^2+\operatorname{Tr}\ln(1-\exp(-4t|\mathcal{M}|)).
\end{aligned} \tag{6.11}$$

So, finally

$$\Gamma_\infty(B)=\tfrac{1}{4}F^2(B)-\operatorname{Tr}\ln(-\nabla^B\nabla^B)+\tfrac{1}{2}\operatorname{Tr}\ln|\mathcal{M}|. \tag{6.12}$$

The formula (6.12) coincides with the standard one-loop formula for the Yang-Mills theory except for the *absolute value* of $\mathcal{M}$. The operator $\mathcal{M}$ has negative eigenvalues (the unstable modes [38], [39]), which led to difficulties in the interpretation of the one-loop effective action (in order to avoid the negative eigenvalues one could introduce an infrared regularisation adding the mass term to $\mathcal{M}$; the mass can be removed after taking the limit $t\to\infty$, we interpret this regularisation independence as infrared stabilisation).

We can compute $\Gamma(A)$ for an Abelian gauge field of constant strength F. We have

$$\begin{aligned}
\operatorname{Tr}\ln|\mathcal{M}|&=\tfrac{1}{2}\operatorname{Tr}\ln\mathcal{M}^2=-\tfrac{1}{2}\operatorname{Tr}\int_0^\infty \mathrm{d}m[(m+\mathrm{i}\mathcal{M})^{-1}+(m-\mathrm{i}\mathcal{M})^{-1}]\\
&=-\tfrac{1}{2}\operatorname{Tr}\int_0^\infty \mathrm{d}m\int_0^\infty \tfrac{1}{2}\,\mathrm{d}s\,\exp(-\tfrac{1}{2}ms)(\exp(\mathrm{i}s\mathcal{M}/2)+\exp(-\mathrm{i}s\mathcal{M}/2))\\
&=4\int_0^\infty\frac{\mathrm{d}s}{s}(2\pi s)^{-2}\left(\frac{u_+s}{\sin u_+s}\,\frac{u_-s}{\sin u_-s}\cos(u_++u_-)s\,\cos(u_+-u_-)s\right.\\
&\qquad\left.-1+\tfrac{5}{12}C_2(G)s^2F^2\theta(1-s)\right)
\end{aligned} \tag{6.13}$$

where

$$u_{\pm}=(C_2(G)/8)^{1/2}([FF+FF^*]^{1/2}\pm[FF-FF^*]^{1/2})$$

with

$$C_2(G)\delta^{av}=f^{abc}f^{vbc} \qquad F^*_{\alpha\beta}=\tfrac{1}{2}\varepsilon_{\alpha\beta\sigma\rho}F_{\sigma\rho}. \tag{6.14}$$

The details of the computation of $\exp(\mathrm{i}s\mathcal{M})$ in (6.13), the renormalisation of the determinants as well as the computation of $\det(-\nabla^B\nabla^B)$ have been discussed in our earlier papers [41]. From the integral representation of $\mathrm{Tr}\ln|\mathcal{M}|$ (6.13) and $\mathrm{Tr}\ln(-\nabla^B\nabla^B)$ [41] we can get the leading behaviour for strong as well as for weak fields

$$\Gamma(B)=\tfrac{1}{8}(2\pi)^{-2}C_2(G)(\tfrac{11}{3})F^2\ln F^2+\mathrm{O}(F^2). \tag{6.15}$$

This behaviour is the same as in the leading logarithm model [42]. Moreover, it can be shown that the effective action (6.12) has a minimum (the gluon condensate) at $F^2\neq 0$. In an infrared regularised theory we get the F^2 behaviour for weak fields and the minimum of Γ at $F^2=0$.

7. Discussion

We have shown that the formulation of the lattice quantum field theory as a classical statistical mechanics by means of a Gibbs measure is equivalent to the stochastic quantisation of Parisi and Wu. The stochastic method shows, however, a remarkable stability with respect to a flip of the sign of the coupling constant. For this reason it appears to be useful in the quantisation of theories needing the charge renormalisation, when we cannot *a priori* impose the requirement of the positivity of the bare charge. The 'wrong sign' of the coupling does not exclude the possibility that the effective action and the Hamiltonian are bounded from below. The sign of the coupling appearing in the perturbation expansion (weak fields) may be different from the behaviour of the effective action for strong fields.

The stochastic quantisation leads to a functional integral in Yang–Mills theory, which has a well defined semiclassical expansion. This also seems to be an outcome of the stability with respect to the flip of the sign of some terms in the action. The unstable modes in the loop expansion are related to the appearance of the infrared Landau ghost. In this case the one-loop propagator $\mathcal{M}^{-1}$ becomes negative at low momenta. This is in contradiction with the positivity of the functional integral. The stochastic method replaces $\mathcal{M}^{-1}$ by $|\mathcal{M}|^{-1}$. From this point of view the stochastic quantisation renders a particular choice of the function, which is to be the sum of the one-loop diagrams ($\mathcal{M}^{-1}$ and $|\mathcal{M}|^{-1}$ have the same perturbation series). However, if we have in view the problem of approximating the functional integral, then $|\mathcal{M}|^{-1}$ is a meaningful approximation, whereas $\mathcal{M}^{-1}$ is not.

The difficulties with the stability of quantum gravity are even more serious. The Einstein action is unbounded from below. Then, like in the Yang–Mills theory the second variation of the action is an operator $\mathcal{M}$ with positive as well as negative eigenvalues. As noted by Greensite and Halpern [6] stochastic quantisation leads to different results from the Hawking method [43]. Quantum gravity results find applications to the thermodynamics of gravitons [43], [44] and in the discussion of gravitational forces at short distances. As in Yang–Mills theory (§ 6), stochastic quantisation replaces the fluctuation operator $\mathcal{M}$ by its absolute value. We shall discuss the behaviour of the effective action in a forthcoming paper [35].

Acknowledgments

The author wishes to thank Professor L Streit and the authorities of ZiF (Bielefeld University) for warm hospitality and the participants of Project No 2 for helpful discussions. He thanks Professor S Kusuoka for illuminating discussions, which have changed decisively § 3.

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